AIAA JOURNAL
Vol. 30, No. 4, April 1992

Three-Dimensional Axisymmetric Vibrations of Orthotropic and Cross-Ply Laminated Hollow Cylinders

Terry D. Hawkes* and Kostas P. Soldatos†
University of Nottingham, Nottingham, NG7 2RD England, United Kingdom

This paper deals with the problem of longitudinal vibrations of homogeneous orthotropic and cross-ply laminated hollow cylinders. The solution of the problem is achieved using a method of successive approximations. The approach is based on the replacement of the hollow cylinder considered with N thin, successive, and coaxial subcylinders. The exact governing equations of each one of these latter fictitious cylinders are replaced by a set of simpler approximate equations that are solved exactly. Connecting all exact solutions obtained by means of appropriate continuity conditions, the corresponding solution of the exact governing equations is successively approached by increasing the value of N. Detailed numerical results dealing with the natural frequencies and vibration mode shapes of homogeneous orthotropic as well as certain two-layered cross-ply cylinders are presented and discussed.

Nomenclature

$\{\bar{C}\}$ = column matrix containing the integration constants C_j (j = 1,2,3,4)
$\{\tilde{C}\}$ = column matrix containing the integration constants $C_j^{(k)}$ (j = 1,2,3,4)
C_{ij} = stiffness elastic constants (i,j = 1,2, . . .,6)
E_L, E_T, G_{LT}, G_{TT}, ν_{LT}, ν_{TT} = engineering elastic constants
$\{F\}$ = column matrix
$\{F^{(k)}\}$ = column matrix
f_u, f_w = functions of z
$f_u^{(k)}$, $f_w^{(k)}$ = functions of $z^{(k)}$
$[G]$ = 4 × 4 matrix
$[G^{(k)}]$ = $4N \times 4N$ matrix
h = thickness of a homogeneous hollow cylinder
$h^{(k)}$ = h/N
L = axial length of a cylinder
$[M]$ = matrix containing the eigenvectors of $[G]$
$[M^{(k)}]$ = matrix containing the eigenvectors of $[G^{(k)}]$
m = axial half-wave number
N = number of subdivisions
R = middle-surface radius of a homogeneous hollow cylinder
$R^{(k)}$ = middle-surface radius of the kth subcylinder
$[T]$ = 4 × 4 matrix
$[\tilde{T}]$ = $4N \times 4N$ matrix
t = time
u, v, w = displacement components
x, s, z = coordinate length parameters
$z^{(k)}$ = transverse coordinate of the kth subcylinder
γ_j = eigenvalues of the matrix $[G]$ (j = 1,2,3,4)
$\gamma_j^{(k)}$ = eigenvalues of the matrix $[G^{(k)}]$ (j = 1,2,3,4)
ϵ_x, ϵ_s, ϵ_z, γ_{sz}, γ_{xz}, γ_{xs} = strain components
ρ = material density
σ_x, σ_s, σ_z, τ_{sz}, τ_{xz}, τ_{xs} = stress components
ω = natural frequency of vibration
ω^* = $(\omega h/\pi L)(\rho/C_{66})^{1/2}$
$\hat{\omega}$ = $\omega L(\rho/E_2)^{1/2}$

Received Oct. 15, 1990; revision received May 29, 1991; accepted for publication June 27, 1991.

*Post Doctoral Research Fellow, Department of Theoretical Mechanics.

†Lecturer, Department of Theoretical Mechanics.

I. Introduction

THE differential equations governing the axisymmetric free vibrations of homogeneous isotropic and orthotropic cylinders are independent of the circumferential cylindrical coordinate parameter. As a result, the axisymmetric vibration problem of such cylinders always splits into two uncoupled problems, that of the torsional and that of the longitudinal vibrations.[1]

The exact, closed-form solution of the torsional vibration problem of homogeneous isotropic hollow cylinders was obtained as a particular case of the general wave propagation solution presented by Gazis.[2] In the case of a homogeneous orthotropic hollow cylinder, the governing differential equation contains two elastic constants and is identical with the differential equation governing the corresponding problem of a transversely isotropic cylinder having its axis in parallel with the axis of material symmetry. Accordingly, the exact, closed-form, torsional solution quoted in the Prasad and Jain[3] analysis, dealing with vibrations of transversely isotropic hollow cylinders, is further applicable in cases of homogeneous orthotropic hollow cylinders (see also Ref. 4).

The corresponding longitudinal vibration problem of homogeneous isotropic hollow cylinders of infinite extent was first considered and solved by Ghosh.[5] The solution of the axisymmetric vibration problem of transversely isotropic hollow cylinders of infinite extent is a particular case of the more general wave propagation solution presented by Mirsky.[6] Based on that exact, closed-form solution, expressed in terms of Bessel and modified Bessel functions of the first and second kind, Mirsky[7] presented numerical results concerning both torsional and longitudinal modes of vibration of such cylinders. Based on Frobenius' method, Mirsky[8] presented a further exact, infinite power series solution of the equations governing the longitudinal vibration problem of infinitely long, orthotropic, solid and hollow cylinders. However, the numerical results presented in Ref. 8 dealt with solid cylinders only.

An extension of the closed-form solutions obtained in Refs. 2 and 3, dealing with torsional vibrations of cross-ply laminated cylinders, is rather obvious. All corresponding solutions obtained, one for each particular layer, must mainly be connected by means of appropriate continuity conditions imposed at the material interfaces of the laminated cylinder considered. In a similar manner, the closed-form solutions presented in Refs. 5 and 6 can also be combined for the study of longitudinal vibrations of laminated cylinders composed of different isotropic and/or transversely isotropic layers. However, for the longitudinal vibration problem of laminated cylinders composed of orthotropic layers, a similar combination of power series solutions, like the ones presented in Ref. 8, is not so evident.

This paper deals with the longitudinal vibration problem of cross-ply laminated hollow cylinders on the basis of the approach presented by Soldatos and Hadjigeorgiou[9] for the free vibration analysis of homogeneous isotropic hollow cylinders and open cylindrical panels. The general method was described by Soong[10] and was used for the buckling analysis of annular plates subjected to arbitrary uniform pressures at the inner and outer edges, for the buckling or vibration analysis of certain one-dimensional problems, and also for the solution of a Mathiew equation.

For the solution of the problem employed, the proposed approach is based on the replacement of the hollow cylinder considered with N thin, successive, and coaxial subcylinders. The exact governing equations of each one of these fictitious cylinders are replaced with a set of simpler approximate equations that are solved exactly. Connecting all exact solutions obtained by means of appropriate continuity conditions, the corresponding solution of the exact governing equations is successively approached by increasing the value of N. The success of the proposed approach is exhibited by means of detailed numerical results dealing with natural frequencies and vibration mode shapes of homogeneous orthotropic as well as certain two-layered cross-ply cylinders.

II. Problem Formulation—Governing Equations

Consider a hollow circular cylinder with an arbitrary constant thickness h and denote with L and R its axial length and its middle-surface radius, respectively. The axial, circumferential, and normal to its middle-surface coordinate length parameters are denoted x, s, and z, respectively, and u, v, and w represent the corresponding displacement components. It is initially assumed that the cylinder considered is constructed of a homogeneous orthotropic linearly elastic material whose principal axes of orthotropy coincide with the axes of the curvilinear coordinate system employed; the case of a cross-ply laminated cylinder will be considered at a later stage of the paper. Accordingly, its elastic behavior is described by Hooke's law,[11,12]

$$\begin{pmatrix}\sigma_x\\ \sigma_s\\ \sigma_z\end{pmatrix}=\begin{pmatrix}C_{11} & C_{12} & C_{13}\\ C_{12} & C_{22} & C_{23}\\ C_{13} & C_{23} & C_{33}\end{pmatrix}\begin{pmatrix}\epsilon_x\\ \epsilon_s\\ \epsilon_z\end{pmatrix}$$
$$(\tau_{sz},\ \tau_{xz},\ \tau_{xs})=(C_{44}\gamma_{sz},\ C_{55}\gamma_{xz},\ C_{66}\gamma_{xs}) \tag{1}$$

The description of the longitudinal vibration problem considered is independent of the circumferential coordinate parameter s. Both u and w are nonzero, but $v=0$. Hence, the three Navier-type governing differential equations (see, i.e., Refs. 13 and 14) are reduced as follows:

$$\begin{gathered}C_{11}u_{,xx}+C_{55}R^{-1}(1+z/R)^{-1}u_{,z}+C_{55}u_{,zz}+(C_{13}+C_{55})w_{,xz}\\ +(C_{12}+C_{55})R^{-1}(1+z/R)^{-1}w_{,x}=\rho u_{,tt}\\ (C_{13}+C_{55})u_{,xz}+(C_{13}-C_{12})R^{-1}u_{,x}+C_{55}w_{,xx}+C_{33}w_{,zz}\\ -C_{22}R^{-2}(1+z/R)^{-2}w+C_{33}R^{-1}(1+z/R)^{-1}w_{,z}=\rho w_{,tt}\end{gathered} \tag{2}$$

The hollow cylinder considered is assumed to be free from external tractions at its lateral surfaces. Hence, requiring the vanishing of transverse stresses ($\sigma_z=\tau_{xz}=0$) at these surfaces, one yields at $z=\mp h/2$:

$$\begin{gathered}C_{13}u_{,x}+C_{23}R^{-1}(1+z/R)^{-1}w+C_{33}w_{,z}=0\\ w_{,x}+u_{,z}=0\end{gathered} \tag{3}$$

Moreover, it is assumed that the cylinder is subjected to the following boundary conditions, imposed at its edges $x=0,L$:

$$\sigma_x=w=0 \tag{4}$$

Because of the appearance of the term $(1+z/R)$, the governing equations (2) are differential equations with variable coefficients. However, for quite thin cylinders ($h/R \ll 1$), a replacement of that term with 1, which is its through thickness average value, is reasonable.[9] Accordingly, Eqs. (2) are replaced by the following differential equations:

$$\begin{gathered}C_{11}u_{,xx}+C_{55}R^{-1}u_{,z}+C_{55}u_{,zz}+(C_{13}+C_{55})w_{,xz}\\ +(C_{12}+C_{55})R^{-1}w_{,x}=\rho u_{,tt}\\ (C_{13}+C_{55})u_{,xz}+(C_{13}-C_{12})R^{-1}u_{,x}+C_{55}w_{,xx}+C_{33}w_{,zz}\\ -C_{22}R^{-2}w+C_{33}R^{-1}w_{,z}=\rho w_{,tt}\end{gathered} \tag{5}$$

Equations (5) have constant coefficients, and for the problem considered, are susceptible to an exact solution. For thin shells, they are considered as a reasonable approximation of the exact governing equations (2). In fact, they may be considered as superior in comparison with the governing equations of any corresponding first-approximation two-dimensional shell theory[15] or its transverse shear deformable analogue.[16,17]

III. Exact Solutions of Equations (5)

For the vibration problem considered, the displacement choice,

$$\begin{aligned}u&=f_u(z)\cos(m\pi x/L)\cos(\omega t)\\ w&=f_w(z)\sin(m\pi x/L)\cos(\omega t)\end{aligned} \tag{6}$$

satisfies exactly the edge boundary conditions (4).

Inserting the displacement choice (6) into Eqs. (5), one obtains the ordinary differential equations,

$$\begin{gathered}C_{55}f_u''+C_{55}R^{-1}f_u'+[C_{11}(m\pi/L)^2-\rho\omega^2]f_u\\ +(C_{13}+C_{55})(m\pi/L)f_w'+(C_{12}+C_{55})(m\pi/L)R^{-1}f_w=0\\ -(m\pi/L)[(C_{13}+C_{55})f_u'+(C_{13}-C_{12})R^{-1}f_u]\\ +C_{33}(f_w''+R^{-1}f_w')-[C_{55}(m\pi/L)^2\\ +C_{22}R^{-2}-\rho\omega^2]f_w=0\end{gathered} \tag{7}$$

where a prime denotes differentiation with respect to z.

The differential eigenvalue problems (7) can be represented in the following form of matrix differential equations:

$$\{F\}'=[G]\{F\},\qquad \{F\}^T=\{f_u,f_u',f_w,f_w'\} \tag{8}$$

where the components of the 4×4 matrix $[G]$ are, in general, dependent on the unknown natural frequency ω; its nonzero components are given in the Appendix. Given an arbitrary value for ω, the eigenvalues γ_j ($j=1,2,3,4$) of $[G]$, as well as the matrix $[M]$ of the corresponding eigenvectors, can be evaluated by using a standard numerical routine. Hence, the solution of Eq. (8) can be expressed as follows:

$$F_i(z)=\sum_{j=1}^{4}[M_{ij}e^{\gamma_j z}]C_j,\qquad (i=1,2,3,4) \tag{9}$$

where C_j are arbitrary constants.

Upon inserting solution (9) into boundary conditions (3), it is found that the following conditions must be satisfied at $z = \mp h/2$:

$$(m\pi/L)C_{13}F_1 - C_{23}(1+z/R)^{-1}R^{-1}F_3 - C_{33}F_4 = 0$$
$$F_2 + (m\pi/L)F_3 = 0 \qquad (10)$$

These conditions in connection with expressions (9) lead to a set of four homogeneous algebraic equations of the form:

$$[T]\{\bar{C}\} = \{0\}, \qquad \{\bar{C}\}^T = \{C_1, C_2, C_3, C_4\} \qquad (11)$$

The components of the matrix $[T]$ are given in the Appendix and are, in general, dependent on the unknown frequency ω. The values of ω that make the determinant of $[T]$ equal to zero, thus providing a nontrivial solution of Eqs. (10), are approximate predictions of the exact frequencies of longitudinal vibration. The associated values of the components of the column matrix $\{\bar{C}\}$ give approximate predictions of the corresponding mode shapes. For the exact solution of the longitudinal vibration problem considered, solution of the governing equations (2) of three-dimensional elasticity is required.

IV. Sequential Solution of Equations (2)

Consider a thick-walled hollow cylinder and imagine it as being composed of N fictitious, successive, and coaxial layers having the same constant thickness h/N and same orthotropic material properties (Fig. 1). Upon choosing a suitably large value of N, each individual layer becomes a thin-walled cylindrical shell, and an approximate solution based on the procedure outlined in the preceding section appears adequate for the study of its dynamic behavior. Connecting next the N solutions obtained by means of appropriate continuity conditions imposed at the fictitious interfaces, and continually increasing N, one can successively approach the solution of the exact equations (2).

For the description of the proposed solution, an individual fictitious layer, say the kth one, is considered. Its axial length and axial coordinate parameters are still L and x, respectively, but its transverse coordinate parameter is denoted with $z^{(k)}$. Moreover, denoting with $h^{(k)}$ and $R^{(k)}$, its thickness and middle surface radius, respectively, the following relations are obtained:

$$h^{(k)} = h/N, \qquad R^{(k)} = R - (N - 2k + 1)h^{(k)}/2$$
$$-h^{(k)}/2 \le z^{(k)} \le h^{(k)}/2 \qquad (12)$$

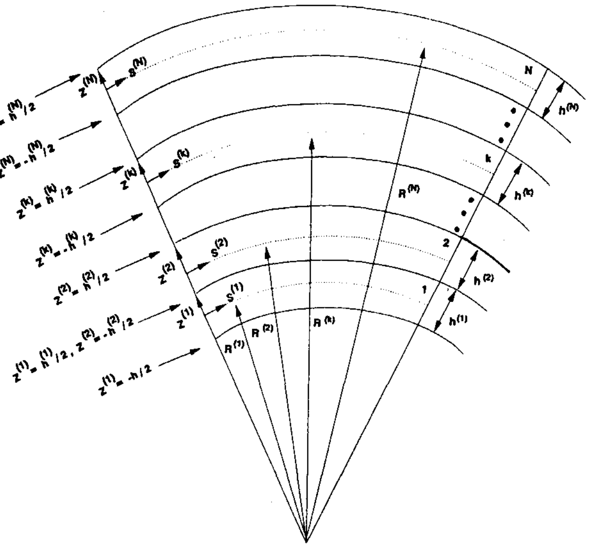

Fig. 1 Nomenclature of a subdivided cylinder.

These parameters should replace all corresponding quantities appearing without a superscript (k) either in the differential equations (5) or wherever needed during the solution procedure outlined in the preceding section; the elastic constants remain unaltered. Moreover, the displacement components appearing in Eqs. (5) should be denoted by $u^{(k)}$ and $w^{(k)}$, whereas their parts dependent on $z^{(k)}$ should be denoted by $f_u^{(k)}$ and $f_w^{(k)}$, respectively.

Accordingly, the differential eigenvalue problem corresponding to Eqs. (8) should be written as follows:

$$\{F^{(k)}\}' = [G^{(k)}]\{F^{(k)}\}, \qquad \{F^{(k)}\}^T = \{f_u^{(k)}, [f_u^{(k)}]', f_w^{(k)}, [f_w^{(k)}]'\} \qquad (13)$$

where a prime denotes differentiation with respect to $z^{(k)}$. Hence, denoting with $\gamma_j^{(k)}$ and $[M^{(k)}]$ the eigenvalues and eigenvectors, respectively, of the matrix $[G^{(k)}]$, the solution of Eqs. (13) can be written in the form:

$$F_i^{(k)}(z) = \sum_{j=1}^{4} \{M_{ij}^{(k)} \exp[\gamma_j^{(k)} z^{(k)}]\} C_j^{(k)}, \qquad (i = 1,2,3,4) \quad (14)$$

where $C_j^{(k)}$ are arbitrary constants.

Requiring free of external tractions lateral surfaces, as well as continuous displacements u and w and transverse stresses σ_z and τ_{xz}, at material fictitious interfaces, the following conditions are obtained ($k = 1,2,\ldots,N-1$):

$$(m\pi/L)C_{13}F_1^{(1)}[-h^{(1)}/2] - C_{23}\{1 - h^{(1)}/[2R^{(1)}]\}^{-1}$$
$$\times F_3^{(1)}[-h^{(1)}/2]/R^{(1)} - C_{33}F_4^{(1)}[-h^{(1)}/2] = 0$$
$$F_2^{(1)}[-h^{(1)}/2] + (m\pi/L)F_3^{(1)}[-h^{(1)}/2] = 0$$
$$F_1^{(k)}[h^{(k)}/2] = F_1^{(k+1)}[-h^{(k+1)}/2]$$
$$F_3^{(k)}[h^{(k)}/2] = F_3^{(k+1)}[-h^{(k+1)}/2]$$
$$(m\pi/L)C_{13}F_1^{(k)}[h^{(k)}/2] - C_{23}\{1 + h^{(k)}/[2R^{(k)}]\}^{-1}$$
$$\times F_3^{(k)}[h^{(k)}/2]/R^{(k)} - C_{33}F_4^{(k)}[h^{(k)}/2]$$
$$= (m\pi/L)C_{13}F_1^{(k+1)}[-h^{(k+1)}/2] \qquad (15)$$
$$-C_{23}\{1 - h^{(k+1)}/[2R^{(k+1)}]\}^{-1}F_3^{(k+1)}[-h^{(k+1)}/2]/R^{(k+1)}$$
$$-C_{33}F_4^{(k+1)}[-h^{(k+1)}/2]$$
$$F_2^{(k)}[h^{(k)}/2] + (m\pi/L)F_3^{(k)}[h^{(k)}/2] = F_2^{(k+1)}[-h^{(k+1)}/2]$$
$$+(m\pi/L)F_3^{(k+1)}[-h^{(k+1)}/2]$$
$$(m\pi/L)C_{13}F_1^{(N)}[h^{(N)}/2] - C_{23}\{1 + h^{(N)}/[2R^{(N)}]\}^{-1}$$
$$\times F_3^{(N)}[h^{(N)}/2]/R^{(N)} - C_{33}F_4^{(N)}[h^{(N)}/2] = 0$$
$$F_2^{(N)}[h^{(N)}/2] + (m\pi/L)F_3^{(N)}[h^{(N)}/2] = 0$$

These conditions together with expressions (14) lead to a system of $4N$ homogeneous algebraic equations, for the $4N$ unknown constants $C_j^{(k)}$ ($j = 1,2,3,4$; $k = 1,2,\ldots,N$), which can be written in the following matrix form:

$$[\tilde{T}]\{\tilde{C}\} = \{0\}$$
$$\{\tilde{C}\}^T = \{C_1^{(1)}, C_2^{(1)}, C_3^{(1)}, C_4^{(1)}, \ldots, C_1^{(N)}, C_2^{(N)}, C_3^{(N)}, C_4^{(N)}\} \qquad (16)$$

The components of the matrix $[\tilde{T}]$ are, in general, dependent on the unknown frequency ω; they may be obtained from the components $[T]$, given in the Appendix, by appending superscripts to those variables that have a dependence on N (see also Ref. 9). The values of ω that make the determinant of $[\bar{T}]$ equal to zero, thus providing a nontrivial solution of Eqs. (16), are approximate predictions of the exact frequencies of longitudinal vibration.

However, as N tends to infinity, the thickness h/N of each fictitious layer tends to zero. It follows that the approximation involved, with the replacement of the term $[1+z^{(k)}/R^{(k)}]$ with 1, tends to become insignificant. Hence, independently of whether the hollow cylinder is a thin- or a thick-walled one, a successive approximation approach is developed that should lead to the exact solution of Eqs. (2) of three-dimensional elasticity. Starting with $N = 1$, one can continually increase the number of layers, thus improving, sequentially, the obtained numerical results. Repetitions then stop at that value of N for which convergence of the obtained results (frequencies and/or mode shapes) is achieved to a desired accuracy.

V. Extension to the Case of Cross-Ply Laminated Cylinders

Unlike Refs. 9 and 14, in which continuity of displacements and their first partial derivatives with respect to the transverse coordinate had been required at all fictitious interfaces, in the present paper continuity of displacements and transverse stresses has been required for the construction of the system of simultaneous algebraic equations (16). Such a mathematically and physically acceptable alteration of continuity conditions does not create any large differences on the rate of convergence of the approach developed. On the contrary, it considerably facilitates the description of its direct extension to consideration of cross-ply laminated hollow cylinders composed of an arbitrary number of specially orthotropic[11] layers.

In addition to the number of fictitious layers that is variable, as being dependent on the rate of convergence of the approach as well as on the desired accuracy of the results obtained, in the case of a cross-ply laminate a given number of real interfaces exists. Such real interfaces separate corresponding real layers composed of different orthotropic materials. Assuming that those real layers are perfectly bonded together, the way of construction of the system of simultaneous algebraic equations (16) is still described by conditions (15).

The only alteration required, in order that the present approach be extended in regard to cross-ply laminated hollow cylinders, deals with the choice of a mathematical formula that will appropriately generate the thicknesses $h^{(k)}$ ($k = 1, 2, \ldots, N$) of the fictitious layers and, in general, will not be the one given in expressions (12). Such a formula is not given explicitly since it is considerably dependent on the type of the particular laminate considered, although, even for a given laminate, its appropriate form may not be unique. Here, it is only mentioned that, after such a mathematical formula has been chosen, the construction of Eqs. (16) will be based on the fact that fictitious interfaces separate layers with same material properties, described with same elastic constants, whereas real interfaces separate layers with different material properties, described by different elastic constants. This distinction is taken into account during the solution procedure described by Eqs. (8) and (9) for each individual layer.

VI. Numerical Results and Discussion

For the longitudinal vibration problem of homogeneous orthotropic cylinders, exact infinite power series solutions have already been presented[8,19] on the basis of the Frobenius method. The solution presented by Nowinski[19] deals with solid cylinders; although the solution presented by Mirsky[8] deals with both solid and hollow cylinders, all numerical results presented in Ref. 8 are also concerned with solid cylinders only. Hence, with respect to longitudinal vibrations of homogeneous and laminated orthotropic hollow cylinders, there are no numerical results in the literature that, based on an alternative exact solution, can be compared with corresponding results obtained on the basis of the present approach.

However, as it was shown in Ref. 9 for the transverse vibration problem of homogeneous isotropic hollow cylinders, all results based on the present approach converge toward the corresponding numerical results that, based on the exact, closed-form solution of Gazis,[2] are tabulated in Armenàkas et al.[18] A restriction of the analysis presented in Secs. III and IV,

Table 1 Influence of N on the prediction of the first six longitudinal vibration frequency parameters ω^* of two-layered antisymmetric cross-ply laminated hollow cylinders [$h/R = 1.0$, $E_{11}^{(1)}/E_{22}^{(1)} = 40$][a]

	Longitudinal vibration frequency parameters					
N	I	II	III	IV	V	VI
2	1.3739	1.5724	2.2594	2.7481	3.2925	4.0525
4	1.4051	1.5851	2.1997	2.7901	3.2827	4.0725
6	1.4136	1.5867	2.1894	2.7983	3.2821	4.0693
8	1.4185	1.5860	2.1855	2.8022	3.2809	4.0690
10	1.4225	1.5846	2.1835	2.8049	3.2792	4.0694
12	1.4262	1.5827	2.1822	2.8072	3.2773	4.0699
14	1.4298	1.5806	2.1812	2.8094	3.2753	4.0704
16	1.4335	1.5783	2.1803	2.8115	3.2732	4.0710

[a]Superscript (1) indicates material properties of the inner (first) real layer of the laminated cylinder considered.

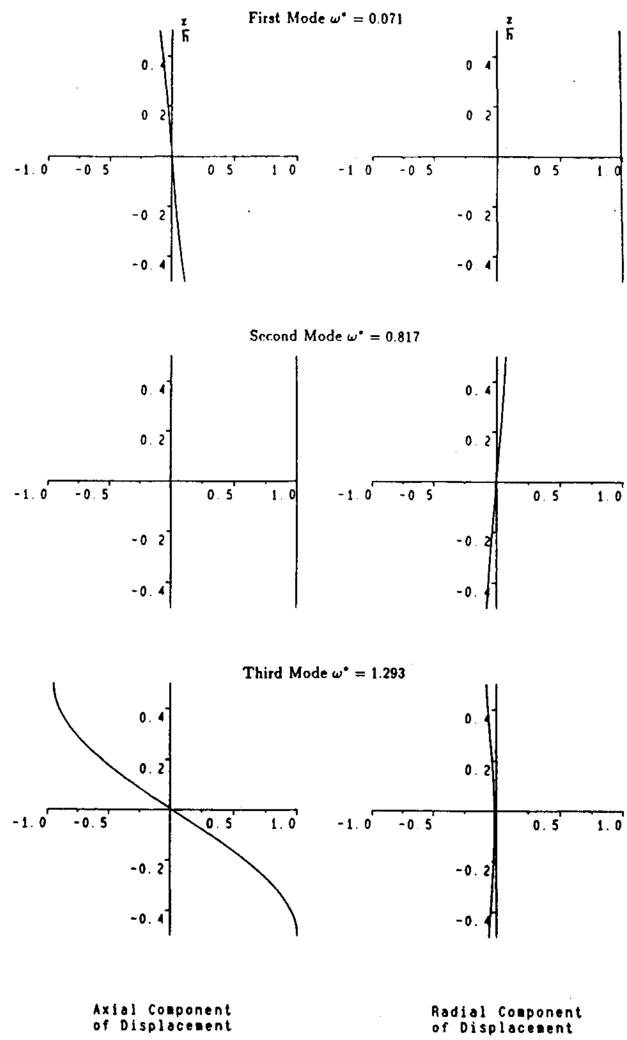

Fig. 2 Frequency parameters ω^* and corresponding mode shapes of homogeneous orthotropic hollow cylinders: $E_{11}/E_{22} = 40$; $h/R = 0.1$; $mR/L = 1$.

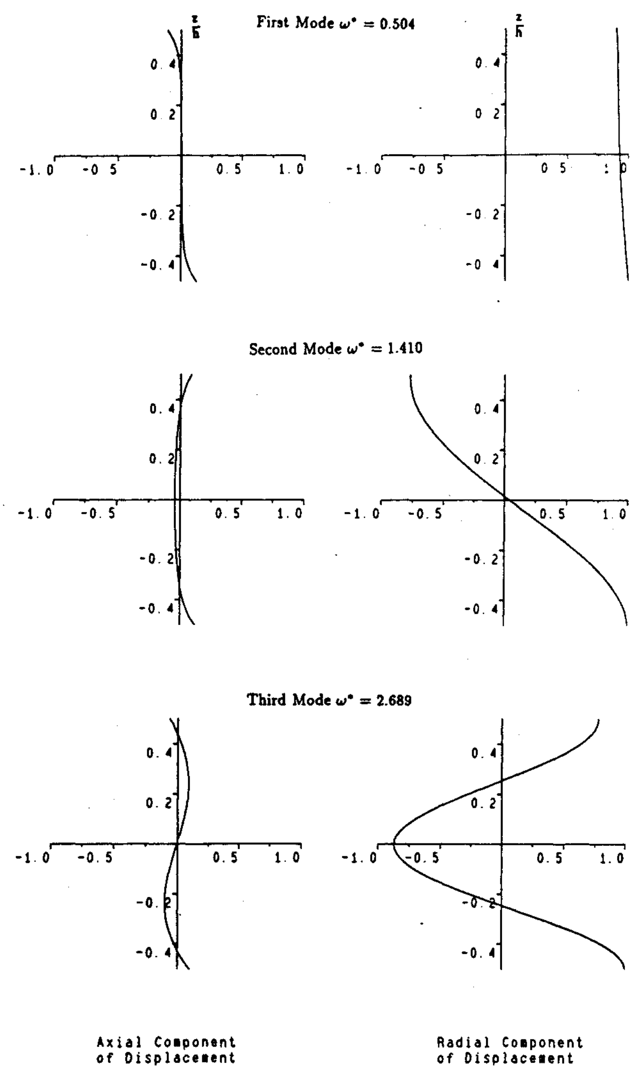

Fig. 3 Frequency parameters ω^* and corresponding mode shapes of homogeneous orthotropic hollow cylinders: $E_{11}/E_{22} = 40$; $h/R = 0.5$; $mR/L = 1$.

dealing with the particular case of isotropic materials, led to the same observation with respect to the longitudinal vibration problem of homogeneous isotropic hollow cylinders. Moreover, as it was shown in Ref. 4 for the torsional vibration problem of homogeneous orthotropic cylinders, all results based on the present approach converge toward the corresponding results obtained on the basis of an alternative exact, closed-form solution.[3] Since the agreement observed between the corresponding results compared in Refs. 4 and 9 was extremely good, it can be concluded that the proposed approach leads, in practice, to the numerical prediction of the exact frequencies of vibration. Such a conclusion is further reinforced from similar numerical comparisons made in Ref. 10 in relation to the mechanical problems considered and solved there.

Since there is an apparent infinite complexity in the class of cross-ply laminates, all results shown in this paper were restricted to cases dealing with homogeneous orthotropic and two-layered antisymmetric cross-ply laminated[11] hollow cylinders: namely, two-ply cylinders composed with layers having the same thicknesses and constructed of the same orthotropic material, but with the material axes of orthotropy alternatively oriented at 0 and 90 deg with respect to the cylindrical axis. In all cases, it was assumed that either the homogeneous cylinders studied or the layers of the two-ply cylinders examined were constructed by a material having the following orthotropic properties:

$$G_{LT}/E_T = 0.6, \quad G_{TT}/E_T = 0.5, \quad \nu_{LT} = \nu_{TT} = 0.25 \qquad (17)$$

with the ratio E_L/E_T being allowed to vary.

For an application of the successive approximation approach described in Sec. V in connection with two-ply laminates, expressions (12), in conjunction with even values of N, were adopted as a mathematical formula generating the thicknesses $h^{(k)}$ ($k = 1,2,\ldots,N$) of the fictitious layers. Accordingly, in each iteration, each one of the two real layers of the cylinder was divided in $N/2$ fictitious sublayers. Then, continuity of displacements and transverse stresses was required at all N fictitious interfaces introduced within each layer as well as on the single real interface of the cylinder considered.

With regard to the longitudinal vibrations of homogeneous orthotropic cylinders, the convergence of the present approach is as fast as the convergence exhibited in Ref. 14, with regard to the transverse vibrations of homogeneous orthotropic cylinders. With regard to laminated cylinders, the fast convergence of the approach is exhibited in Table 1,

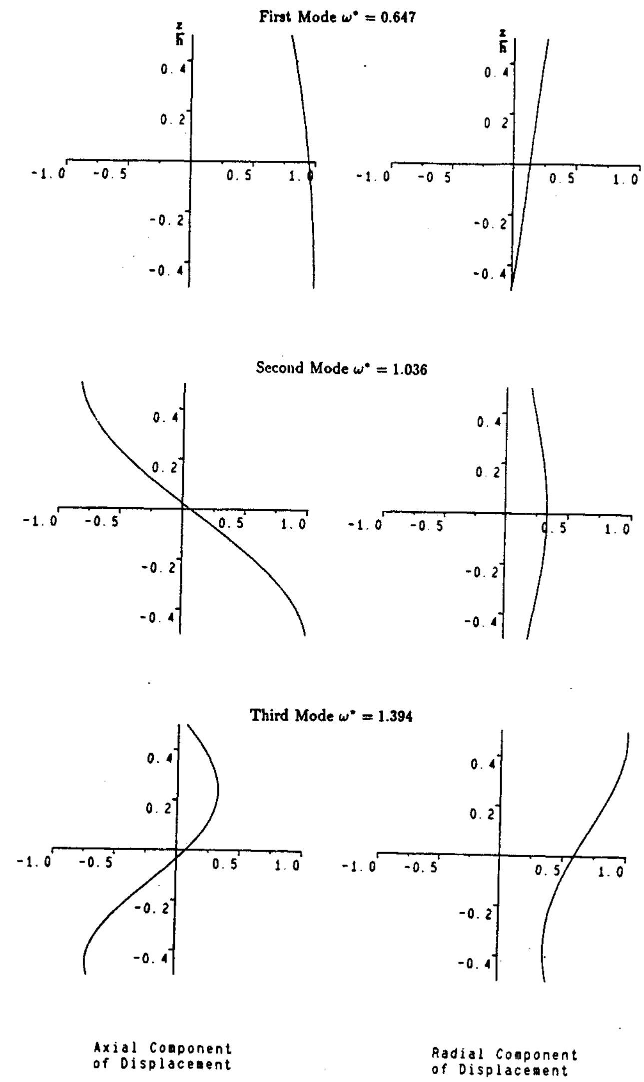

Fig. 4 Frequency parameters ω^* and corresponding mode shapes of homogeneous orthotropic hollow cylinders: $E_{22}/E_{11} = 40$; $h/R = 0.5$; $mR/L = 1$.

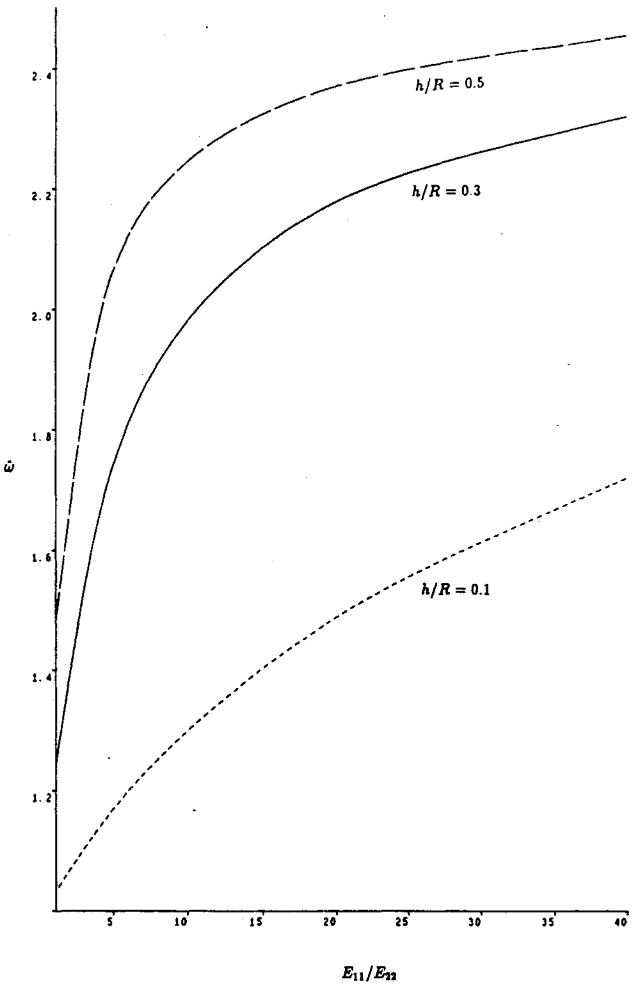

Fig. 5 Lowest frequency parameter $\hat{\omega}$ as a function of the stiffness ratio E_{11}/E_{22} for homogeneous orthotropic hollow cylinders: $mR/L = 1$.

where the influence of the number of fictitious layers N is shown on the prediction of the first six longitudinal vibration frequency parameters,

$$\omega^* = \frac{\omega h}{\pi L} (\rho / C_{66})^{1/2} \tag{18}$$

of a two-ply graphite-epoxy cylinder having thickness-to-middle-surface ratio $h/R = 1.0$. In accordance with the rates of convergence shown in Ref. 14 as well as in Table 1, 8 successive approximations for homogeneous cylinders and 16 for two-ply laminated ones were always found enough to guarantee for the desirable convergence of both the frequencies and mode shapes that are shown next.

Figures 2 show the mode shapes corresponding to the first three longitudinal frequency parameters ω^* of a relatively thin ($h/R = 0.1$) homogeneous hollow cylinder composed of a highly orthotropic graphite-epoxy material ($E_{11}/E_{22} = 40$); the fibers are oriented in parallel to the axis of the cylinder. A comparison with the corresponding results presented in Armenàkas et al.[18] (p. 181) for a homogeneous isotropic cylinder shows that, due to the thinness of the cylinder, orthotropicity has not considerably affected the mode shapes of the first three longitudinal frequencies of vibration. The displacement component that is predominantly influenced by the motion looks very similar for both the orthotropic and the isotropic cylinders.

Different observations can be made in Figs. 3, where corresponding results are shown for a much thicker ($h/R = 0.5$) hollow cylinder composed of the same graphite-epoxy material ($E_{11}/E_{22} = 40$). It is apparent that the predominant influence of the motion associated with all three first frequencies of vibration is mainly radial. The strong limiting effect of the orthotropic material, which is highly inextensible in the axial direction of the cylinder, can be seen in the axial components of all modes shown. In particular, in the lowest mode, there is effectively no axial displacement within the interior of the cylinder, with a small displacement within a boundary layer near the inner and outer surfaces of the cylinder.

Figures 4 show the mode shapes corresponding to the first three longitudinal frequency parameters ω^* of the same thick ($h/R = 0.5$) homogeneous hollow cylinder composed of the same graphite-epoxy material, but with the principal material direction switched from the axial to the circumferential direction ($E_{22}/E_{11} = 40$). It can be seen immediately that the constraint on the axial displacement component observed in Figs. 3 is not present, as should be the case. Indeed, the axial displacement component is dominant in the lowest two modes, and it is considerably large even in the third mode in which the radial displacement component is the dominant one.

For several ratios of thickness-to-middle-surface radius h/R, Fig. 5 shows the variation of the lowest longitudinal frequency parameter,

$$\hat{\omega} = \omega L (\rho / E_2)^{1/2} \tag{19}$$

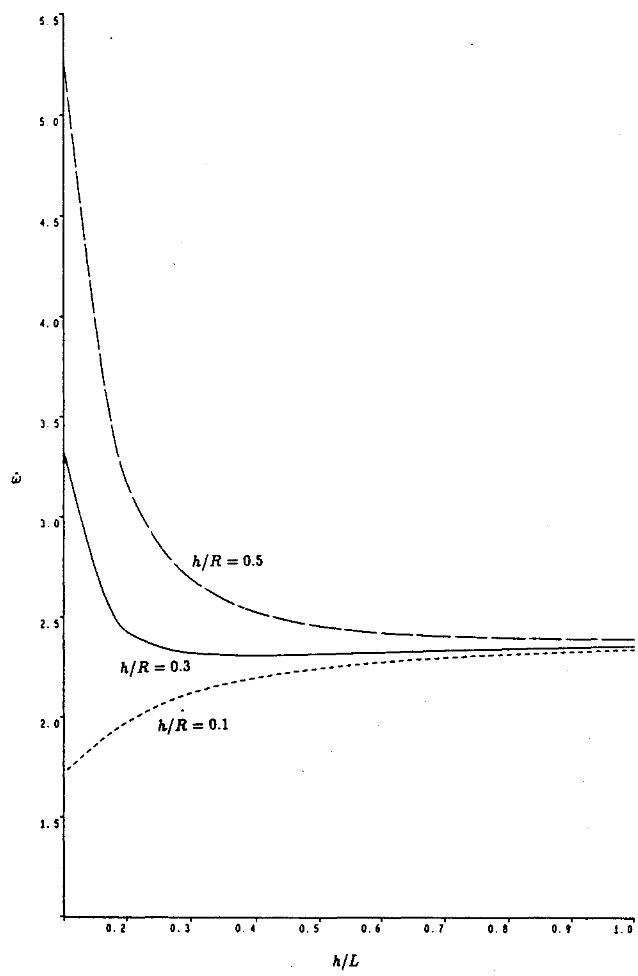

Fig. 6 Lowest frequency parameter $\hat{\omega}$ as a function of the thickness-to-axial-length ratio h/L for homogeneous orthotropic hollow cylinders: $E_{11}/E_{22} = 40$; $m = 1$.

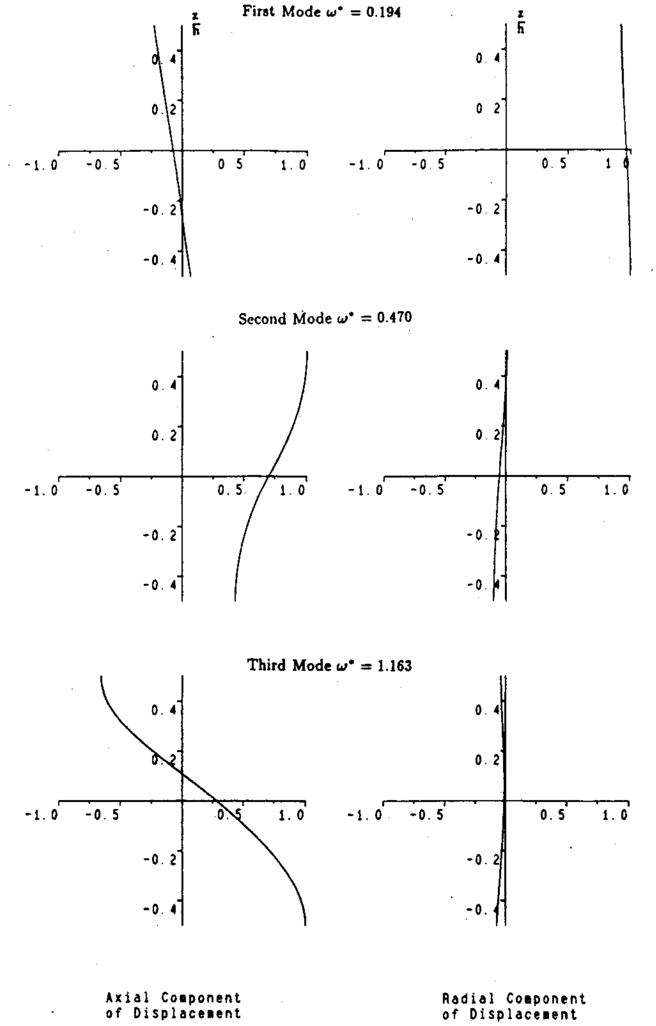

Fig. 7 Frequency parameters ω^* and corresponding mode shapes of two-layered cross-ply laminated hollow cylinders: $E_{11}^{(1)}/E_{22}^{(1)} = 40$; $h/R = 0.1$; $mR/L = 1$.

as a function of the stiffness ratio E_{11}/E_{22}. It is apparent that the lowest longitudinal frequency increases with increasing either the stiffness, in the axial direction, or the thickness of the cylinder. It can further be seen that the rate of increase in $\hat{\omega}$ decreases with either increasing E_{11}/E_{22} or h/R. This occurs as the cylinder becomes increasingly inextensible in the axial direction with increase in either E_{11}/E_{22} or h/R.

For several ratios of thickness-to-middle-surface radius h/R, Fig. 6 shows the variation of the lowest longitudinal frequency parameter $\hat{\omega}$ as a function of the thickness-to-the-axial-length ratio of a homogeneous orthotropic hollow cylinder composed of a graphite-epoxy material ($E_{11}/E_{22} = 40$). It can be seen that for cylinders with small thickness-to-length ratios ($h/L =$ 0.1–0.2), the lowest frequency differs widely with differing thickness-to-radius ratio. However, upon increasing h/L, the frequencies appear to tend asymptotically to some fixed value, which is independent of the thickness-to-radius ratio and occurs as the lowest longitudinal frequency of a zero length cylinder ($h/L \to \infty$).

For the results shown next, dealing with two-ply cylinders, the inner layer of the cylinder is always considered as the basic or first ply. Hence, all elastic constants referring to the inner layer are denoted either without a superscript or with a superscript (1), whereas elastic constants referring to the outer layer are always denoted with a superscript (2).

Figures 7 show the mode shapes corresponding to the first three longitudinal frequency parameters ω^* of a relatively thin ($h/R = 0.1$) two-layered hollow cylinder composed of highly orthotropic graphite-epoxy layers; the inner layer is reinforced across its axial direction $[E_{11}^{(1)}/E_{22}^{(1)} = E_{22}^{(2)}/E_{11}^{(2)} = 40]$. It is apparent that the mode shapes of the laminated cylinder correspond closely with the mode shapes shown in Figs. 2 for the corresponding homogeneous cylinder, with the only significant difference appearing in the second mode, which is the lowest among the predominantly axial modes. In the homogeneous cylinder (Figs. 2), its axial displacement component is about constant through the thickness, whereas the effect of the change in fiber direction along $z/h = 0$ causes a marked change in the two-ply cylinder (Figs. 7). Because of the much higher axial reinforcement, the axial displacement is considerably diminished in the inner layer of the cylinder.

Figures 8 show the mode shapes corresponding to the first three longitudinal frequency parameters ω^* of a thick ($h/R =$ 0.5) two-layered cylinder composed of graphite-epoxy layers; again, the inner layer is reinforced across its axial direction $[E_{11}^{(1)}/E_{22}^{(1)} = E_{22}^{(2)}/E_{11}^{(2)} = 40]$. A comparison with the corresponding results presented in Figs. 3 shows that, due to the much higher axial reinforcement, the axial displacements are very small in the interior of the inner ply. However, the boundary-layer zone has been enlarged considerably near the interface $z/h = 0$ since the outer ply has been reinforced circumferentially. In fact, in all three modes shown, the axial displacement component is considerably large throughout the outer ply of the cylinder whereas, unlike in Figs. 3, the second mode of longitudinal vibration is a predominantly axial one. Moreover, although the first mode of vibration is still a predominantly radial mode, the magnitude of its axial displace-

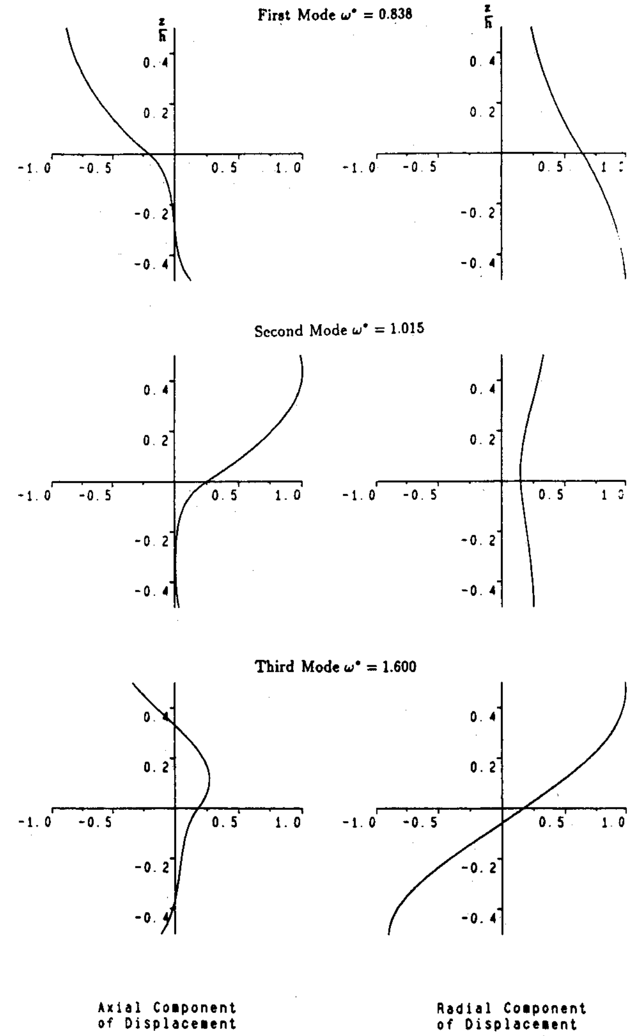

Fig. 8 Frequency parameters ω^* and corresponding mode shapes of two-layered cross-ply laminated hollow cylinders: $E_{11}^{(1)}/E_{22}^{(1)} = 40$; $h/R = 0.5$; $mR/L = 1$.

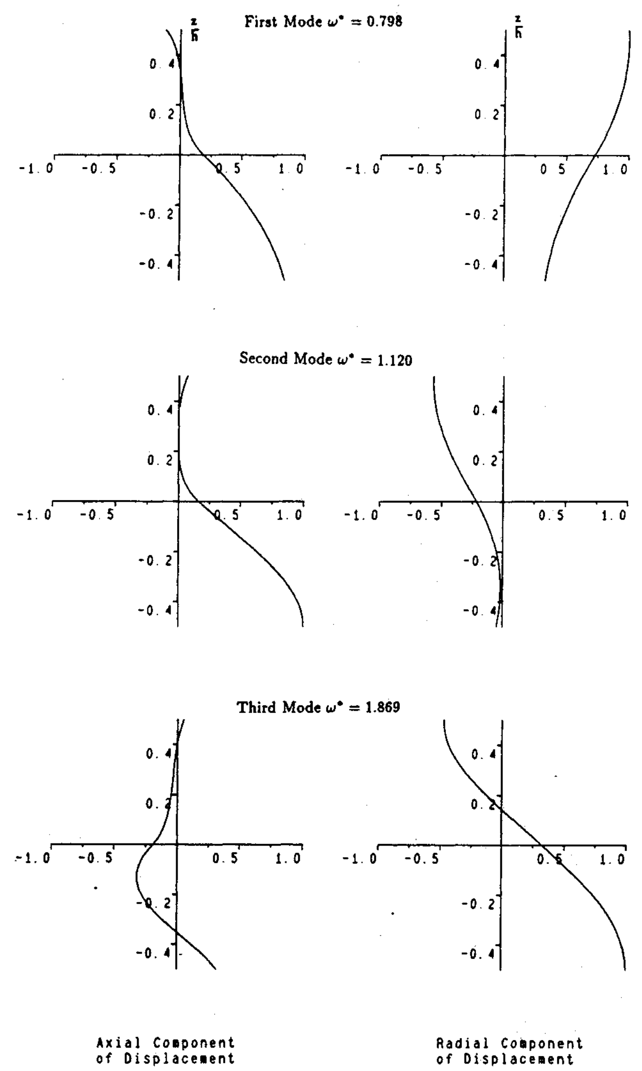

Fig. 9 Frequency parameters ω^* and corresponding mode shapes of two-layered cross-ply laminated hollow cylinders: $E_{22}^{(1)}/E_{11}^{(1)} = 40$; $h/R = 0.5$; $mR/L = 1$.

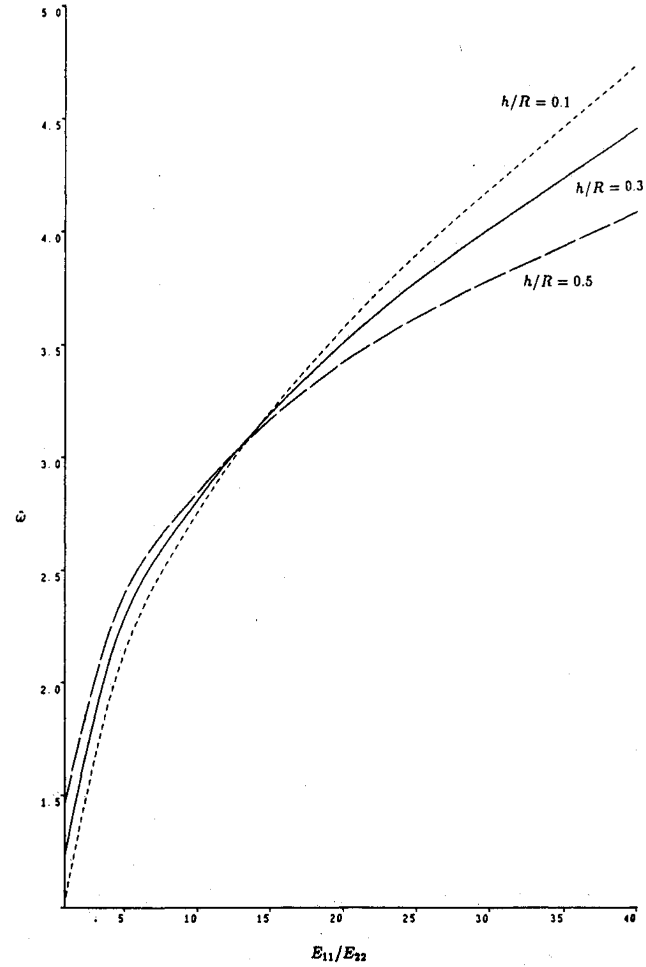

Fig. 10 Lowest frequency parameter $\hat{\omega}$ as a function of the stiffness ratio $E_{11}^{(1)}/E_{22}^{(1)}$ for two-layered cross-ply laminated hollow cylinders: $mR/L = 1$.

ment on the top surface of the cylinder has become very much comparable with the magnitude of its radial displacement on the inner cylindrical surface.

These latter observations are not unexpected since the mode shapes shown in Figs. 8 could be considered as a combination of the mode shapes for a material highly constrained in the axial direction within the inner ply, as in Figs. 3, and one that is highly constrained in the circumferential direction, as in Figs. 4. This can also be seen in Figs. 9, where similar results are shown for a corresponding thick ($h/R = 0.5$) graphite-epoxy two-ply cylinder having its inner layer reinforced across its circumference $[E_{22}^{(1)}/E_{11}^{(1)} = E_{11}^{(2)}/E_{22}^{(2)} = 40]$. As expected, the laminate is highly constrained in the axial direction within the outer ply, with little constraint in the inner ply. In fact, it can be seen that the behavior within the inner ply in Figs. 9 is almost identical to the behavior within the outer ply in Figs. 8 and, similarly, the displacements within the outer ply in Figs. 9 are almost identical to the displacements within the inner ply in Figs. 8. The small differences are due to the differences in the radius of curvature of the inner and outer layers in each case.

For several ratios of thickness-to-middle-surface radius h/R of a two-ply cylinder, with the inner layer reinforced across its axial direction $[E_{11}^{(1)}/E_{22}^{(1)} = E_{22}^{(2)}/E_{11}^{(2)}]$, Fig. 10 shows the variation of the lowest longitudinal frequency parameter $\hat{\omega}$ as a function of the stiffness ratio. It is apparent that, as in Fig. 5 for homogeneous hollow cylinders, the lowest frequency increases with increasing stiffness. However, due to the well-known coupling effect between bending and extension, occurring in laminated composite structures, unlike the results drawn in Fig. 5 for homogeneous cylinders, the lowest longitudinal frequency of the two-ply cylinder is not always increasing with increasing the thickness of the cylinder. Although this is initially the case, there is a limiting value of the stiffness ratio $[E_{11}^{(1)}/E_{22}^{(1)} \simeq 14]$ beyond which the lowest frequency is decreasing with increasing the thickness-to-middle-surface-radius ratio.

For several ratios of thickness-to-middle-surface radius h/R of a two-ply graphite-epoxy cylinder, with the inner layer reinforced across its axial direction $[E_{11}^{(1)}/E_{22}^{(1)} = E_{22}^{(2)}/E_{11}^{(2)} = 40]$, Fig. 11 shows the variation of the lowest longitudinal frequency parameter $\hat{\omega}$ as a function of the thickness-to-axial-length ratio. Because of the lamination coupling effect between bending and extension, the behavior of the results drawn in Fig. 11 appear quite different from the behavior of the corresponding results drawn in Fig. 6 for homogeneous cylinders, especially for the relatively thinner cylinder ($h/R = 0.1$). However, upon increasing h/L, the frequencies still appear to tend asymptotically to some fixed value, which is independent of h/R and occurs as the lowest longitudinal frequency of a zero length cylinder ($h/L \to \infty$).

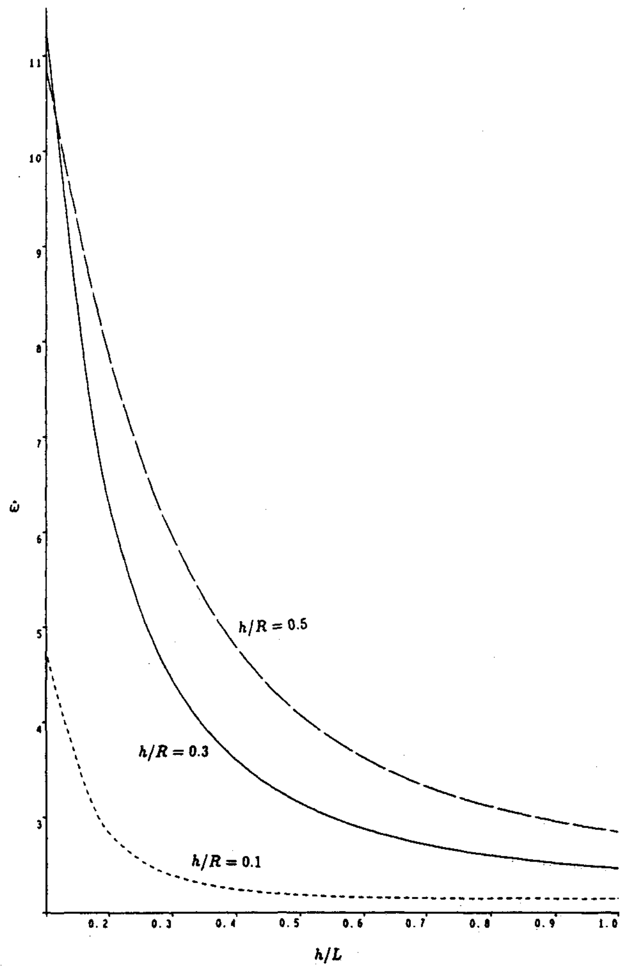

Fig. 11 Lowest frequency parameter $\hat{\omega}$ as a function of the thickness-to-axial-length ratio h/L for two-layered cross-ply laminated hollow cylinders: $E_{11}^{(1)}/E_{22}^{(1)} = 40$; $m = 1$.

VII. Concluding Remarks

A successive approximation method suitable for three-dimensional axisymmetric vibration analysis of homogeneous orthotropic and cross-ply laminated hollow cylinders of arbitrary thickness was presented. In certain particular cases, the efficiency of the proposed method was checked by comparing its results with corresponding numerical results based on alternative exact, closed-form solutions available in the field literature.[2,4,9,18] Moreover, a fast convergence of corresponding numerical results, dealing with the longitudinal vibration problem of homogeneous and two-layered antisymmetric cross-ply laminated cylinders was observed.

Further applications of the new approach concerning the study of more complicated problems related with the mechanical behavior of laminated composite cylinders are visible. For instance, an extension of the method with regard to the axisymmetric dynamic behavior of cylinders composed of a more complicated stacking arrangement (antisymmetric or unsymmetric angle-ply laminates) is evident. One has only to replace the displacement model that corresponds to the model (6) with a displacement model that satisfies the edge boundary conditions of the particular problem considered. Moreover, in applications dealing with the general dynamic behavior of angle-ply laminates, the part of the displacement model that depends on the circumferential cylindrical coordinate (see, i.e., Refs. 9 and 14) can be replaced with the exponential complex function proposed in Ref. 20, in connection with two-dimensional shell-type governing differential equations.

On the other hand, the achievements of the present approach are certainly based on the replacement of the exact three-dimensional governing differential equations, which possess variable coefficients, with a number of corresponding equations with constant coefficients that can be solved, successively, with an analytical technique suitable for the solution of the corresponding flat-plate problem. This observation leads toward the conclusion that the proposed approach is directly applicable to different classes of problems dealing with the mechanical behavior of homogeneous and laminated composite cylinders, such as static problems in linear elasticity or static and dynamic nonlinear problems.

Appendix: [G] and [T] Matrices

The nonzero components of the matrix $[G]$ appearing in Eqs. (8) are as follows:

$$\begin{aligned}
G_{12} &= G_{34} = 1 \\
G_{21} &= [C_{11}(m\pi/L)^2 - \rho\omega^2]/C_{55} \\
G_{22} &= G_{44} = -R^{-1} \\
G_{41} &= (C_{13} - C_{12})(m\pi/L)R^{-1}/C_{33} \\
G_{42} &= (C_{13} + C_{55})(m\pi/L)/C_{33} \\
G_{43} &= [C_{55}(m\pi/L)^2 + C_{22}R^{-2} - \rho\omega^2]/C_{33}
\end{aligned} \tag{A1}$$

The components of the matrix $[T]$ appearing in Eqs. (11) are given as follows ($j = 1,2,3,4$):

$$\begin{aligned}
T_{1j} &= \{C_{13}m\pi M_{1j}/L - R^{-1}[1 - h/(2R)]^{-1}M_{3j} \\
&\quad - C_{33}M_{4j}\}e^{-\gamma_j h/2} \\
T_{2j} &= [M_{2j} + m\pi M_{3j}/L]e^{-\gamma_j h/2} \\
T_{3j} &= \{C_{13}m\pi M_{1j}/L - R^{-1}[1 + h/(2R)]^{-1}M_{3j} \\
&\quad - C_{33}M_{4j}\}e^{\gamma_j h/2} \\
T_{4j} &= [M_{2j} + m\pi M_{3j}/L]e^{\gamma_j h/2}
\end{aligned} \tag{A2}$$

Acknowledgments

The financial support provided by the Science and Engineering Research Council is acknowledged. We would like to thank our colleagues in the Department of Theoretical Mechanics, Nottingham University, in which this study was carried out during a one-year sabbatical awarded to Kostas P. Soldatos from the University of Ioannina, Greece. We are grateful to G. A. Thurston, Associate Editor of *AIAA Journal,* who drew our attention to Ref. 10 during the review process of this paper.

References

[1]Love, A. E. H., *A Treatise on the Mathematical Theory of Elasticity,* 4th ed., Cambridge Univ. Press, New York, 1952, pp. 285-292.

[2]Gazis, D. C., "Three-Dimensional Investigation of the Propagation of Waves in Hollow Circular Cylinders, I. Analytical Foundation," *Journal of the Acoustical Society of America,* Vol. 31, No. 5, 1959, pp. 568-573.

[3]Prasad, C., and Jain, R. K., "Vibrations of Transversely Isotropic Cylindrical Shells of Finite Length," *Journal of the Acoustical Society of America,* Vol. 38, No. 6, 1965, pp. 1006-1009.

[4]Soldatos, K. P., "An Iterative Solution of Bessel Equation Based on Torsional Vibrations of Orthotropic Hollow Cylinders," *Journal of Sound and Vibration,* Vol. 151, No. 1, 1991, pp. 149-152.

[5]Ghosh, J., "Longitudinal Vibrations of a Hollow Cylinder," *Bulletin of the Calcutta Mathematical Society,* Vol. 14, 1923, pp. 31-40.

[6]Mirsky, I., "Wave Propagation in Transversely Isotropic Circular Cylinders. Part I: Theory," *Journal of the Acoustical Society of America,* Vol. 37, No. 6, 1965, pp. 1016–1021.

[7]Mirsky, I., "Wave Propagation in Transversely Isotropic Circular Cylinders. Part II: Numerical Results," *Journal of the Acoustical Society of America,* Vol. 37, No. 6, 1965, pp. 1022–1026.

[8]Mirsky, I., "Axisymmetric Vibrations of Orthotropic Cylinders," *Journal of the Acoustical Society of America,* Vol. 36, No. 11, 1964, pp. 2106–2112.

[9]Soldatos, K. P., and Hadjigeorgiou, V. P., "Three-Dimensional Solution of the Free Vibration Problem of Homogeneous Isotropic Cylindrical Shells and Panels," *Journal of Sound and Vibration,* Vol. 137, No. 3, 1990, pp. 369–384.

[10]Soong, T.-V., "A Subdivisional Method for Linear Systems," *Proceedings of the 11th AIAA/ASME Structures, Structural Dynamics, and Materials Conference,* 1970, pp. 211–223.

[11]Jones, R. M., *Mechanics of Composite Material,* McGraw-Hill, New York, 1975, pp. 34, 35.

[12]Whitney, J. M., *Structural Analysis of Laminated Anisotropic Plates,* Technomic, Lancaster, 1987, pp. 6–8.

[13]Chou, F. H., and Achenbach, J. D., "Three-Dimensional Vibrations of Orthotropic Cylinders," *Journal of the Engineering Mechanics Division,* Vol. 98, Aug. 1972, pp. 813–822.

[14]Soldatos, K. P., and Hawkes, T. D., "An Iterative Three-Dimensional Solution of the Free Vibration Problem of Homogeneous Orthotropic Hollow Cylinders," *Structural Dynamics,* edited by W. B. Krätzig et al., Vol. 2, Balkema, Rotterdam, The Netherlands, 1991, pp. 883–889.

[15]Soldatos, K. P., "A Comparison of Some Shell Theories Used for the Dynamic Analysis of Cross-ply Laminated Circular Cylindrical Panels," *Journal of Sound and Vibration,* Vol. 97, No. 2, 1984, pp. 305–319.

[16]Soldatos, K. P., "On the Theories Used for the Wave Propagation in Laminated Composite Thin Elastic Shells," *ZAMP,* Vol. 36, 1985, pp. 120–133.

[17]Soldatos, K. P., "Influence of Thickness Shear Deformation on the Free Vibrations of Rectangular Plates, Cylindrical Panels and Cylinders of Antisymmetric Angle-ply Construction," *Journal of Sound and Vibration,* Vol. 119, No. 1, 1987, pp. 111–137.

[18]Armenàkas, A., Gazis, D. C., and Herrmann, G., *Free Vibrations of Circular Cylindrical Shells,* Pergamon, Oxford, England, UK, 1969.

[19]Nowinski, J. L., "Propagation of Longitudinal Waves in Circular Cylindrically Orthotropic Bars," *Journal of Engineering for Industry,* Vol. 89, No. 3, 1967, pp. 408–412.

[20]Soldatos, K. P., "A Refined Laminated Plate and Shell Theory with Applications," *Journal of Sound and Vibration,* Vol. 144, No. 1, 1991, pp. 109–129.